Splittings of π_*BTOP

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The goal is to look at [Bru68, Thm 1.4] for a few examples. Numerics at manifold atlas and wikipedia. Recall that $\pi_n(PL/O) \cong \Theta_n$ (due to Mazur-Hirsch) and we have the exact sequence

$$0 \to bP_{n+1} \to \Theta_n \to \pi_n^S/\mathrm{Im}(J)$$

now in degree $n = 4n - 1$ Brumfiel's [Thm 1.3] shows that this sequence is short exact (continues to the right with a zero) and moreover split (really the splitting the exactness was due to KM), that is

$$\Theta_{4n-1} \cong bP_{4n} \oplus \pi_{4n-1}^S/\mathrm{Im}(J)$$

Apparently it is again due to Mazur-Hirsch that the fibration

$$PL/O \to BO \to BPL$$

has its long exact sequence break into short exact sequences

$$0 \to \pi_{n+1}(BO) \to \pi_{n+1}(BPL) \to \pi_n(PL/O) \to 0$$

Looking at this sequence for -1 modulo 4 case we see

$$0 \to \pi_{4n}(BO) \to \pi_{4n}(BPL) \to \pi_{4n-1}(PL/O) \to 0$$

using Mazur-Hirsch and Bott periodicity

$$0 \to \mathbb{Z} \to \pi_{4n+1}(BPL) \to \Theta_{4n-1} \to 0$$

and finally Brumfiels theorem

$$0 \to \mathbb{Z} \to \pi_{4n+1}(BPL) \to bP_{4n} \oplus \pi_{4n-1}^S/\mathrm{Im}(J) \to 0$$

Analysing the splitting of this sequences Brumfiel's [Thm 1.4] is that for $n > 2$

$$\pi_{4n}(BPL) \cong \mathbb{Z} \oplus \pi_{4n-1}^S/\mathrm{Im}(J)'$$

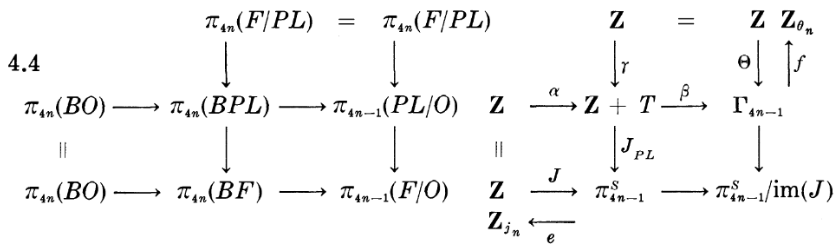
where $\mathrm{Im}(J)' \subseteq \mathrm{Im}(J)$ is the subgroup of elements of odd order. Note that this amounts to showing that there is a splitting and that the first map $\mathbb{Z} \to \pi_{4n+1}(BPL)$ has its image isomorphic to $bP_{4n} \oplus \mathrm{Im}(J)'/\mathrm{Im}(J)$ (using the first iso theorem on the second map, then exactness, then the third iso theorem on the second summand):::: IS that true?. Using that $\left(\pi_{4n-1}^S/\mathrm{Im}(J)'\right)/\left(\mathrm{Im}(J)/\mathrm{Im}(J)'\right) \cong \pi_{4n-1}^S/\mathrm{Im}(J)$.

1 Proof Sketch

Let $\Sigma \in \Theta_{4n-1}$, then they show in §3 that it bounds two different manifolds, one spin manifold and one 'U' manifold. Lets look at just the n odd case, the even case just differs slightly with an annoying introduction of a factor of two. Taking the index of these bounded manifolds is an invariant group homomorphism of the homotopy sphere that lands in the cyclic group of order $\theta_n = 8|bP_{4n}|$. They then show that in fact the index itself is also divided by 8 (of the homotopy sphere). Thus the index of these bounded manifolds defines a group homomorphism to a cyclic group of order θ_n (def 1.2).

The proof of [1.3] is in [Thm 4.3]. Take a generator of bP_{4n}, then they show that it bounds a manifold with index 1. This implies that the index restricts to an isomorphism on bP_{4n}. This index provides the splitting to the exact sequence.

For [1.4] he combines [Thm 4.6] and [Thm 4.7] which describe the even and odd tosion of $\pi_{4n}(BPL)$ respectively. First he summarises the diagrams that he is considering as



T is the torsion subgroup. The Pontryagin class of a bundle over S^{4n} defines a non-trivial homomorphism $\pi_{4n}(BPL) \to \mathbb{Q}$ (a homotopy class, defines a PL bundle then take cohomology etc), because the codomain is torsion free and the map is non-trivial (therefore injective on the $\mathbb{Z}$ component) we can conclude that $T = \ker(p_n)$.

Lemma. *$\gamma(1) = n + t$ implies that $T \not\subseteq \ker J_{PL}$ for some $n \neq 0$.*

Proof. Basically we just want to see why J_{PL} injects T into the codomain. The image of γ is multiples $k\gamma(1)$ and hence $kn + kt$. Using the definition of a coproduct we are interested in the map $J_{PL}(0, t)$ and hence when the image of γ is *completely torsion*, but because γ sends a generator to a non-zero element this can never happen (other than for 0) and so no torsion elements are in the kernel.

Now [Thm 4.6] shows that J_{PL} is a two primary isomorphism. Using the fact that J_{PL} is an injection on torsion (above) and the verticle ses of the diagram we can produce a new ses by modding out by torsion. The image of γ was computed on $\mathbb{Z}$ to be the odd parts of the denominator of Bernoulli numbers. Using the first isomorphim theorem we see that modding out the homotopy group of spheres by the torsion we get a group that is just the odd torsion of the homotopy group of spheres, thus we must have modded out by the even torsion. I think there is something im missing like the fact that odd torsion in homotopy groups are related to den Bn? Ptherwise it seems I might be killing more...

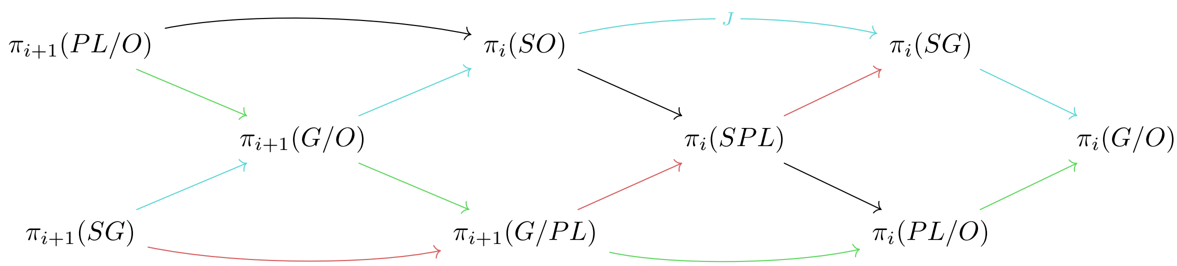
Finally the odd torsion is shown to be isomorphic to the odd torsion of the kernel of the index invariant (defined above) or the odd torsion of the kernel of the Adams e invariant inside Θ_{4n-1} and π^S_{4n-1} respectively. Uses A hat genus... Not very familiar. Use the fact that the pontryagin classes are zero (from above) to conclude some things about the index.

2 KM Braid

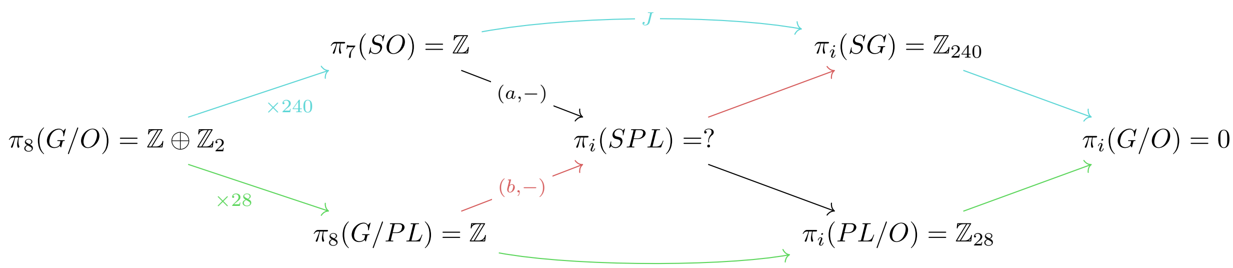
Lets look at another way of resolving this problem, of finding what $\pi_i BPL$. There is fibration

$$SO \to SPL \to SG$$

which generates three LES by taking pairs of spaces and looking at the LES for the pair, and a fourth LES by weaving them. This is summarised in the diagram, where the different colors indicated different exact sequences



Consider the case for $i = 7$,



The zero on the far right tells us that the last maps are surjective. We claim that $\pi_i(SPL) \cong \mathbb{Z} \oplus \mathbb{Z}_{\gcd(a,b)}$. One can check this by tensoring at the primes not appearing in the GCD, using right exactness and seeing that the torsion vanishes...

Proof.

The zero on the far right was very helpful. In higher homotopy groups this will not vanish. Notice that if we resolve this section of the LES into its SES parts then we get the KM SES. Thus what Brumfield first does is resolves that extension problem and then applies the same idea to compute the middle groups.

3 $n = 3$

For $n = 3$ the sequence becomes

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_{13}(BPL) \longrightarrow bP_{12} \oplus \pi_{11}^S/\mathrm{Im}(J_{11}) \longrightarrow 0$$
$$bP_{12} \oplus \pi_{11}^S/\mathrm{Im}(J_{11}) \cong \mathbb{Z}_{992} \oplus 0$$

Using the table on wikipedia we have that $\pi_{11}^S = \mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_7$ and $\mathrm{Im}(J_{11})' = \mathbb{Z}_9 \oplus \mathbb{Z}_7$ hence in this case the theorem asserts that $\pi_{13}(BPL) \cong \mathbb{Z} \oplus \mathbb{Z}_8$.

4 $n = 4$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_{17}(BPL) \longrightarrow bP_{16} \oplus \pi_{15}^S/\mathrm{Im}(J_{15}) \longrightarrow 0$$
$$bP_{16} \oplus \pi_{15}^S/\mathrm{Im}(J_{15}) \cong \mathbb{Z}_{8128} \oplus \mathbb{Z}_2$$

Using the table on wikipedia we have that $\pi_{15}^S = 32 \cdot 2 \cdot 3 \cdot 5$ and $\mathrm{Im}(J_{15})' = 3 \cdot 5$ hence in this case the theorem asserts that $\pi_{17}(BPL) \cong \mathbb{Z} \oplus (32 \cdot 2)$.

5 $n = 5$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_{21}(BPL) & \longrightarrow & bP_{20} \oplus \pi_{19}^S/\mathrm{Im}(J_{19}) & \longrightarrow & 0 \\ & & & & & & \wr\| & & \\ & & & & & & \mathbb{Z}_{130816} \oplus \mathbb{Z}_2 & & \end{array}$$

Using the table on wikipedia we have that $\pi_{19}^S = 8 \cdot 2 \cdot 3 \cdot 11$ and $\mathrm{Im}(J_{19})' = 3 \cdot 11$ hence in this case the theorem asserts that $\pi_{21}(BPL) \cong \mathbb{Z} \oplus 8 \cdot 2$.

6 $n = 6$

This is the first time there is non-two torsion coker$J$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_{25}(BPL) & \longrightarrow & bP_{24} \oplus \pi_{23}^S/\mathrm{Im}(J_{23}) & \longrightarrow & 0 \\ & & & & & & \wr\| & & \\ & & & & & & \mathbb{Z}_{297040170} \oplus \left(\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_3\right) & & \end{array}$$

Using the table on wikipedia we have that $\pi_{23}^S = 16 \cdot 8 \cdot 2 \cdot 9 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ and $\mathrm{Im}(J_{23})' = 9 \cdot 5 \cdot 7 \cdot 13$ hence in this case the theorem asserts that $\pi_{21}(BPL) \cong \mathbb{Z} \oplus 16 \cdot 8 \cdot 2 \cdot 3$.

References

[Bru68] G. Brumfiel. On the Homotopy Groups of BPL and PL/O. *The Annals of Mathematics*, 88(2):291, September 1968.